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Decay of the Two-Point Function in One-Dimensional $O(N)$ Spin Models with Long-Range Interactions

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Using Griffiths and Lieb–Simon type inequalities, it is shown that the two-point function of ferromagnetic spin models with N components in one dimension decays like the interaction $J(n) \sim n^{-\gamma}$ provided that $1 \leqslant N \leqslant 4$ and $T > T_c$.

KEY WORDS:

1. INTRODUCTION AND MAIN RESULT

As is well known, classical spin models with a continuous symmetry in two dimensions lead to scale invariant field theories with the nonlinear sigma-model action

$$\frac{1}{2T} \int d^2x(\nabla \mathbf{s}(\mathbf{x}))^2 \tag{1.1}$$

where $\mathbf{s}$ is a unit spin with N components. The resulting behaviour distinguishes between an abelian, $N=2$ plane rotor, and a non-abelian symmetry group, $N \geqslant 3$. In the latter case the two-point correlation, $g(\mathbf{x}) = \langle \mathbf{s}(\mathbf{x}) \cdot \mathbf{s}(0) \rangle$ decays exponentially at any finite temperature with a finite correlation length $\xi(T) \sim \exp[2\pi/(N-2)T]$.[1,2] On the other hand for an XY-symmetry, the exponential decay holds only above the Kosterlitz–Thouless critical temperature T_{KT}. The phase at low temperatures exhibits

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power-law decay $g(\mathbf{x}) \sim |\mathbf{x}|^{-\eta(T)}$ with a continuously varying exponent $\eta(T)$.[3] Qualitatively this behaviour can be understood through the spin-wave approximation

$$(\nabla \mathbf{s})^2 \approx (\nabla \varphi)^2 \tag{1.2}$$

with $\mathbf{s} = (\cos \varphi, \sin \varphi)$, neglecting the periodicity of the phase variable φ. Since the approximate action is Gaussian, the correlation function $g(\mathbf{x}) = \Re \langle \exp[i(\varphi(\mathbf{x}) - \varphi(0))] \rangle$ can be calculated easily, yielding the power law decay $|\mathbf{x}|^{-\eta(T)}$ with $\eta = T/2\pi$. The vortex excitations lead, at low T, only to a finite renormalisation of η.[3]

In one-dimensional models the spatial dimension can be mimicked by a long range interaction with a decay as $|n|^{-\gamma}$, n being a point on the one-dimensional lattice. One notices that for $\gamma = 2$ the action is again scale invariant. For this marginal case, it was conjectured early by Thouless[4] that the $N = 1$ Ising model has a spontaneous magnetization m^* below a nonzero critical temperature T_c. The spontaneous magnetization jumps to a finite value at T_c, yet the transition is continuous. This conclusion was confirmed through an analysis of the equivalent Kondo problem[5, 6] and later proven rigorously.[7, 8] A renormalization group calculation for long range spin models in one dimension both for $N = 1$ and a continuous symmetry was performed by Kosterlitz.[9] Within a one loop calculation he showed that there is always a low temperature spontaneous magnetization provided $1 < \gamma < 2$ ($\gamma > 1$ being necessary to have an extensive free energy). In the marginal case $\gamma = 2$ and for $N \geqslant 2$, the associated beta function vanishes quadratically near the trivial fixed point $T = 0$. This indicates that $T_c = 0$ for $N \geqslant 2$ and an exponential behaviour $\chi(T) \sim \exp[2\pi^2/(N-1)T]$ for $T \to 0$ of the susceptibility. Due to the power law interaction there can be no finite correlation length, however.

In our present note we discuss the long distance behaviour of the two-point function $g(n)$. In the phase where $m^* = 0$ it is shown that if $g(n)$ has a power law decay at all, it is necessarily equal to that of the interaction. In particular, a spin wave approximation is qualitatively incorrect for long range models even at very low temperatures. Moreover, in the XY-case with marginal $\gamma = 2$, our rigorous bounds rule out the appearance of a low temperature phase with a continuously varying exponent $\eta(T)$ and infinite susceptibility, which has been claimed in the literature on the basis of spin wave theory and Monte Carlo simulations.[10–12] Our result is of direct relevance to the problem of strong tunneling in the so called single electron box, showing that the Coulomb blockade at zero temperature is not destroyed even for large conductance.[13]

To be more precise, we consider the spin Hamiltonian

$$H = -\tfrac{1}{2} \sum_{m,n} J(m-n)\, \mathbf{s}_m \cdot \mathbf{s}_n, \tag{1.3}$$

where the couplings are ferromagnetic, $J(n) \geqslant 0$, and decay as

$$J(n) \cong |n|^{-\gamma} \tag{1.4}$$

for $|n| \to \infty$. As before the two-point function is defined by $g(n) = \langle \mathbf{s}_n \cdot \mathbf{s}_0 \rangle$ in the infinite volume limit with free boundary conditions. Then

$$\lim_{n \to \infty} g(n) = m^{*2}, \tag{1.5}$$

where m^* is the spontaneous magnetization with the standard convention that $m^* = 0$ for $T > T_c$ and $m^* > 0$ for $T < T_c$. We define the scaling exponent η by

$$g(n) - m^{*2} \cong |n|^{-\eta} \tag{1.6}$$

for large n. The magnetic susceptibility is given by

$$\chi = \frac{\beta}{N} \sum_n (g(n) - m^{*2}). \tag{1.7}$$

If $\eta < 1$, then $\chi = \infty$.

The qualitative phase diagram for such ferromagnetic models is rather well understood. For $\gamma > 2$ one has $m^* = 0$ and for $\gamma < 2$ one has $m^* > 0$ at sufficiently low temperatures. In the marginal case, $\gamma = 2$, the number of components becomes relevant. Whereas for the Ising model, $N = 1$, $m^* > 0$ at low T,[7] for $N \geqslant 2$ Simon[15] proves that $m^* = 0$ at any finite T. The decay of the two-point function has been studied on a rigorous level mostly for the Ising model with particular attention to the marginal case $\gamma = 2$:[14] For $T > T_c$ one has $\eta = 2$. At T_c m^* jumps to a non-zero value, the Thouless effect, and η jumps to zero. Below T_c, $\eta(T)$ increases with decreasing T and locks to its high temperature value $\eta = 2$ at some critical value $T_c^* < T_c$. Although η varies continuously, the overall behaviour is obviously quite distinct from the standard Kosterlitz–Thouless scenario in the short range $d = 2$, $N = 2$ case.

Here we show that if $N = 1, 2, 3, 4$, if $m^* = 0$, and if $g(n)$ is known to have some decay already, then $\eta = \gamma$. A lower bound of this form is known from Griffiths second inequality for arbitrary N.[16] A corresponding upper bound is slightly more involved. It uses a Lieb–Simon type inequality,[17, 18], which relies on Gaussian domination of the four-point function. Although this is expected to hold in general, it has been proved only for $N=1, 2, 3, 4$ components.[19]

In the following section we give the details of the argument. In fact, it would be of interest to have a numerical solution of the nonlinear integral equation (2.8), which could be used as a sharp test of Monte-Carlo simulations.

2. BOUNDS ON THE TWO-POINT FUNCTION

We consider ferromagnetic spin models with N components in one space dimension with Hamiltonian (1.3). Here $\mathbf{s}_n$ is the N component spin at lattice site n, n integer, with $|\mathbf{s}_n|=1$. The couplings J satisfy $J(n)\geqslant 0$, $J(n)=J(-n)$ and have the asymptotic decay (1.4). To have an extensive free energy we require $\gamma>1$. The equilibrium distribution in finite volume $[-\ell,..., \ell]$ is given by

$$Z^{-1}\exp[-\beta H]\prod_{n=-\ell}^{\ell}\delta(|\mathbf{s}_n|-1)\,d^N s_n. \tag{2.1}$$

We choose free boundary conditions, i.e., $\mathbf{s}_n=0$ for n outside $[-\ell,..., \ell]$ and denote the corresponding expectation by $\langle\cdot\rangle_\ell$. The two-point function in the infinite volume limit $\ell\to\infty$ is then defined by

$$g(m-n)=\langle\mathbf{s}_m\cdot\mathbf{s}_n\rangle=\lim_{\ell\to\infty}\langle\mathbf{s}_m\cdot\mathbf{s}_n\rangle_\ell\geqslant 0 \tag{2.2}$$

If $m^*=0$, g is independent of the boundary conditions.

To discuss the asymptotic decay of g we first note that by Griffiths second inequality g is increasing in the couplings. Thus

$$g(n)\geqslant\frac{1}{Z}\int\delta(|\mathbf{s}_0|-1)\,d^N s_0\delta(|\mathbf{s}_n|-1)\,d^N\mathbf{s}_n\exp[\beta J(n)\,\mathbf{s}_0\cdot\mathbf{s}_n]\,\mathbf{s}_0\cdot\mathbf{s}_n \tag{2.3}$$

which proves that $g(n)$ cannot decrease *faster* than the couplings $J(n)$.

The *upper* bound for g is slightly more complicated and uses the well known Lieb–Simon type inequality. We define

$$\Lambda_L=\{u, v\mid\text{either }|u|\leqslant L,\,|v|>L\text{ or }|u|>L,\,|v|\leqslant L\} \tag{2.4}$$

and split the Hamiltonian as

$$H_\lambda = H_1 + \lambda H_2$$

$$H_1 = -\tfrac{1}{2} \sum_{\substack{m,\, n = -\ell \\ m,\, n \in \Lambda_L^c}}^{\ell} J(m-n)\, \mathbf{s}_m \cdot \mathbf{s}_n, \tag{2.5}$$

$$H_2 = -\tfrac{1}{2} \sum_{\substack{m,\, n = -\ell \\ m,\, n \in \Lambda_L}}^{\ell} J(m-n)\, \mathbf{s}_m \cdot \mathbf{s}_n.$$

Differentiating with respect to λ we obtain

$$\begin{aligned}\langle \mathbf{s}_m \cdot \mathbf{s}_n \rangle_\ell &= \langle \mathbf{s}_m \cdot \mathbf{s}_n \rangle_{\ell,\, \lambda=0} + \int_0^1 d\lambda \frac{d}{d\lambda} \langle \mathbf{s}_m \cdot \mathbf{s}_n \rangle_{\ell,\, \lambda} \\ &= \langle \mathbf{s}_m \cdot \mathbf{s}_n \rangle_{\ell,\, \lambda=0} + \int_0^1 d\lambda \frac{\beta}{2} \sum_{\substack{u,\, v = -\ell \\ u,\, v \in \Lambda_L}}^{\ell} J(u-v) \\ &\quad \times (\langle (\mathbf{s}_m \cdot \mathbf{s}_n)(\mathbf{s}_u \cdot \mathbf{s}_v) \rangle_{\ell,\, \lambda} - \langle \mathbf{s}_m \cdot \mathbf{s}_n \rangle_{\ell,\, \lambda} \langle \mathbf{s}_u \cdot \mathbf{s}_v \rangle_{\ell,\, \lambda})\end{aligned} \tag{2.6}$$

We choose $|m| \leqslant L$, $|n| > L$. Then the first term in (2.6) vanishes. For the second term we use the Gaussian domination valid for $N = 1, 2, 3, 4$ [(19]

$$\begin{aligned}&\langle (\mathbf{s}_m \cdot \mathbf{s}_n)(\mathbf{s}_u \cdot \mathbf{s}_v) \rangle_{\ell,\, \lambda} \\ &\quad \leqslant \langle \mathbf{s}_m \cdot \mathbf{s}_n \rangle_{\ell,\, \lambda} \langle \mathbf{s}_u \cdot \mathbf{s}_v \rangle_{\ell,\, \lambda} \\ &\qquad + \frac{1}{N} (\langle \mathbf{s}_m \cdot \mathbf{s}_u \rangle_{\ell,\, \lambda} \langle \mathbf{s}_m \cdot \mathbf{s}_v \rangle_{\ell,\, \lambda} + \langle \mathbf{s}_m \cdot \mathbf{s}_v \rangle_{\ell,\, \lambda} \langle \mathbf{s}_n \cdot \mathbf{s}_u \rangle_{\ell,\, \lambda})\end{aligned} \tag{2.7}$$

and set $\lambda = 1$ because $\langle \mathbf{s}_m \cdot \mathbf{s}_n \rangle_{\ell,\, \lambda}$ is increasing in λ. Finally we take $\ell \to \infty$ and arrive at

$$g(n) \leqslant (\beta/N) \sum_{|u| \leqslant L} \sum_{|v| > L} g(u)\, J(u-v)\, g(v-n) \tag{2.8}$$

for $|n| > L$.

The integral inequality (2.8) is studied in [20]. In essence, one splits the v-sum into terms with $|v| \leqslant |n/2|$ and those with $|v| > |n/2|$. This yields for $|n| > \hat{L}$, $\hat{L}$ fixed,

$$g(n) \leqslant c'\, |n|^{-\gamma} + \alpha(n)\, g(n/2) \tag{2.9}$$

with $\alpha(n) \to 0$ as $|n| \to \infty$. Iterating (2.9) results in a bound as

$$g(n) \leqslant c\,|n|^{-\gamma} \tag{2.10}$$

The precise conditions, cf. ref. 20, Lemma 5.4, for the validity of (2.10) are (i) for $\gamma > 2$ it is required that $\lim_{n\to\infty} g(n) = 0$, which we know already from $m^* = 0$, (ii) for $\gamma = 2$ it is required that $g(n) \leqslant c_1(1 + \log(1 + |n|))^{-1}$ with a suitable constant c_1 depending on the prefactor of J, (iii) for $1 < \gamma < 2$ it is required that $g(n) \leqslant c_2(1 + |n|)^{\gamma - 2}$ with a suitable constant c_2 depending on the prefactor of J. We conclude that under the stated conditions on $g(n)$ and if $N = 1, 2, 3, 4$, $m^* = 0$, then

$$g(n) \simeq \text{const.}\ |n|^{-\gamma} \tag{2.11}$$

for large $|n|$.

For Ising spins the bounds (2.3), (2.10) have recently been sharpened,[21] such as to determine also the prefactor in (2.11). Generalizing to the present case, we conjecture that

$$\lim_{n\to\infty} \frac{1}{\beta J(n)} \langle \mathbf{s}_0 \cdot \mathbf{s}_n \rangle = \frac{1}{\beta^2} N \chi^2 \tag{2.12}$$

The proof in ref. 21 uses the FK and percolation representation for the lower bound and the random current representation for the upper bound, which unfortunately are special to $N = 1$.

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